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Cognitive Models: The Missing Link to Learning Fraction Multiplication and Division

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This quasi-experimental study aims to streamline cognitive models on fraction multiplication and division that contain the most worthwhile features of other existing models. Its exploratory nature and its approach to proof elicitation can be used to help establish its effectiveness in building students' understanding of fractions as compared to the traditional algorithmic way of teaching, vis-à-vis the students' negative notions about learning fractions. Interestingly, the study showed the benefits and drawbacks of using these cognitive models in the teaching and learning of mathematics.

Key words: cognitive models, fraction multiplication and division, instructional intervention

Introduction

Over the years, the teaching of fractions continues to attract the attention of mathematics teachers and education researchers worldwide (Cramer, 2002; Freiman & Volkov, 2004). Long standing debates as to whether it has to be introduced as counting or as a form of measurement, or whether it represents procedural, factual or conceptual knowledge is relative to the success in learning fractions (Meagher, 2002; Johnson & Koedinger, 2001). To solve fraction multiplication, students are traditionally taught the cancellation algorithm (cancel-and-multiply), while for fraction division, students are asked to follow the computational procedure of inverting the divisor and changing the operation to multiplication (invert-and-multiply). However, many students are unable to correct their errors and clear up their confusion due to a lack of understanding of the underlying rationale in fraction multiplication and division (NCTM 2000; de Castro, 2004; Tirosh, 2000). This was manifested by the alarming results of the Third International Mathematics and Science Study – Repeat (TIMSS-R), where the Philippine sample's performance on fractions had a mean score of 378 against the 487 international average, indicating the performance of our students on fractions far below international standards (Ibe, 2001; TIMSS-R, 2000). The purpose of this study is to streamline cognitive models on fraction multiplication and division so as to narrow down the most worthwhile features of other existing models and to establish its effectiveness in building students' understanding of fractions.

Students' Initial Concept of Fractions

Students, in their early years, have varied initial concepts of fractions. Stafylidou and Vosniadou (2004) assert that children develop their numerical value of

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fractions through assimilation of this new information into their existing conceptual structures of natural numbers, thus assuming that the same algorithms hold true in performing the fundamental operations on rational numbers as in natural numbers.

Moss and Case (1999) suggested that in the realm of rational numbers, children have two natural schema: one global structure for proportional evaluation and one numerical structure of splitting/doubling. Hunting's (1999) study of five-year old children focused on early conceptions of fractional quantities and suggested that there is considerable evidence to support the idea of "one half" as being well established in children's mathematical schema at an early age. He argues that this and other knowledge about subdivision of quantities forming what he calls "prefraction knowledge" can be drawn upon to help students develop more formal notion of fractions from a very early age. Mack (1998) stressed the importance of drawing on students' informal knowledge. She used equal sharing situations in which parts of a part can be used to develop a basis for understanding multiplication of fractions such as sharing half a pizza equally among three children results in a child getting one third of one half. This showed that students did not think of taking a part in terms of multiplication but that their strong experience with the concept could be developed later (Meagher, 2002).

Tzur (1999), for his part, interpreted children's initial reorganization of fraction conception as falling into three strands: equidivision of wholes into parts, recursive partitioning of parts (splitting) and reconstruction of the unit (i.e. the whole). He further suggested that teachers consider one of these strands at a time in teaching rational numbers.

Problems Encountered in Teaching Fraction Multiplication and Division

Previous research exercises have identified major problems with current teaching methods in the area of fractions. The first deals with a syntactic (rules) rather than semantic (meaning) emphasis of teaching rational numbers, wherein teachers often emphasize technical procedures in doing fraction operations at the expense of developing a strong sense in children of the meaning of rational numbers (Moss & Case, 1999; Lubinski & Fox, 1998). This problem led to algorithmically-based mistakes, which result when an algorithm is viewed as a meaningless series of steps so that students often forget some of these steps or change them in ways that lead to errors (Tirosh, 2000; Freiman & Volkov, 2004).

Secondly, teachers often take an adult-centered rather than a child-centered approach, emphasizing a fully formed adult conception of rational numbers, not taking into consideration their schema and informal knowledge of fractions, thus denying children a spontaneous means of learning fractions (Moss & Case, 1999). One of the reasons pointed out as to why the mathematical notion of fractions is systematically misinterpreted is because fractions are not consistent with the counting principles that apply to natural numbers to which children often relate (Stafylidou & Vosniadou, 2004) (see Table 1). They further concluded that early knowledge about natural numbers may, in fact, serve as a barrier to learning about fractions, given children's constructivist tendency to distort new information (about fractions) to fit their counting based number theory. Tirosh (2000) refers to these as intuitively-based mistakes which stem from the predominance of the partitive model used with natural numbers wherein children argue that it is impossible to solve division expressions with a dividend smaller than the divisor.

A third issue deals with the limited formal knowledge on fractions. Students count the number of shaded parts in a figure and the total number of parts so that each part is regarded as an independent entity or amount (Moss & Case, 1999). Yoshida and Sawano (2002) referred to these points

Table 1.

Differences between Fundamental Operations of Natural Numbers and Fractions

Operation	Natural Number	Fraction
Addition and Subtraction	Supported by the natural number's sequence	Not supported by the natural number's sequence
Multiplication	Product is larger than the factors	Product may either be higher or lower than the factors
Division	Quotient is smaller than the dividend	Quotient may either be higher or lower than the dividend

as cognitive obstacles that make the learning of fractions difficult for the students, such as the concept of equal partitioning and the invariance of the whole. Although students acquire the knowledge of equal partitioning informally before learning fractions, they had difficulty relating it to their formal knowledge of fractions. The representation of a fraction as a number less than one, magnitudes of one as a whole should be of the same size for all fractions. Students find difficulty reconciling this idea once they deal with different models of rational numbers (Tirosh, 2000). In the case of middle school students, this confusion comes from the lack of knowledge when it comes to deciding rules in choosing the best fraction representation from three possible basic models that would fit the problem requirements, namely: the area or regional model, the length or measurement model and the set model (Parmar, 2003).

Finally considerable problems in the use of a notation can also act as a hindrance to students' development which centered on teachers' perception (Moss & Case, 1999).

New Teaching Approaches

NCTM (2000) Standards offer a variety of ways to improve initial fraction instruction for early elementary years. Instruction must give focus to developing an understanding of the meaning of the symbols, examining relationships and building initial concepts of order and equivalence in fractions. Conceptual understanding should be developed before conceptual fluency, since fluency in fraction computations will be dealt with in the latter years. Since many children experience difficulty in constructing the idea of specific fractions, NCTM also suggested the use of physical objects, diagrams and real-world situations and that instruction should help students make connections from these representations to verbal meanings and symbols.

A Rational Number Project (RNP) Curriculum (Cramer et al., 2002), involving elementary students, emphasized the use of multiple physical models and translations within and between modes of representations, such as pictorial, manipulative, verbal, real-world and symbolic representations. Students approached the fundamental operations tasks conceptually by building on their constructed mental images of fractions.

Some teachers make use of paper folding to represent fractions in lieu of pie charts. Streefland (1991) made use of real life situations to develop children's understanding of rational numbers. Moss and Case's (1999) own approach started with beakers filled with various levels of water and asked students to label beakers from 1 to 100 based on their fullness or emptiness. This approach produced deeper, more proportionally based understanding of rational numbers and the addition and subtraction processes. There was greater emphasis on meaning (semantics) over procedures, on the proportional nature of fractions highlighting differences between the integers and rational numbers, on children's natural methods of solving problems and the use of alternative forms of visual representation as a mediator between proportional quantities and numerical representations, that is an alternative to the use of pie charts.

Comparing the benefits and drawbacks of the traditional method of teaching fraction multiplication and division and the above-mentioned strategies, neither of these strategies proved to be strong on all of the five dimensions taken into consideration in choosing an instructional method: difficulty in learning, efficiency, generality, retention and transfer (Johnson & Koedinger, 2001). There had been trade-offs for learning each strategy which allow for an informal decision on whether such strategy will be taught and how it should be taught.

The Present Study

On the basis that students had already acquired basic implicit knowledge of fraction representation, equivalence and order, and that students had an informal knowledge of partitioning, cognitive models were streamlined to help students understand multiplication and division of fractions, culminating with an explanation of why when we multiply fractions, the product is usually lesser than the factors. Most teachers only understand this notion instrumentally, that is they know how to apply the procedure without understanding why the procedure works (de Castro, 2004). People at all levels can work with fractions but only a few could provide a conceptual explanation for many of the procedures associated with fractions. The topic of fractions is an example of the notion that given any topic, everyone understands something and no one understands everything and therefore a key to effective instruction is to find out what knowledge students possess and to build on that

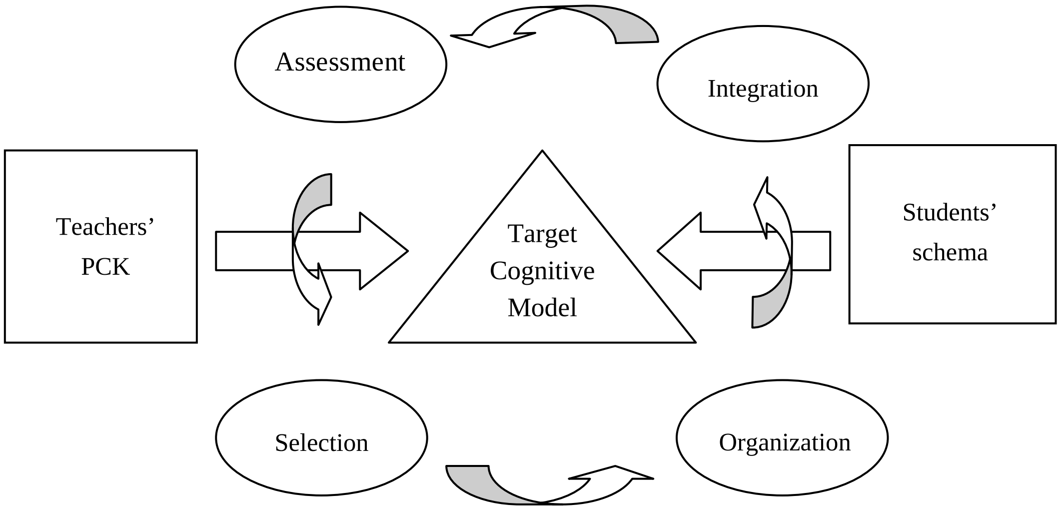


Figure 1. Development process of a cognitive model in mathematics

knowledge.

The purpose of this paper was to consider the potential of the use of cognitive models in making students understand difficult subject matter such as fraction multiplication and division. For the purpose of this study, students who were low achievers and who had prior knowledge of the subject matter taken into consideration were taken as respondents of the study. To help students replace their misconceptions with scientifically correct knowledge, the cognitive models present students with the basic line of reasoning underlying the correct interpretation of the phenomena that are the base of their misconception. Knowing the correct line of reasoning enables the student to self-explain the phenomenon, which according to Chi (1996) in the study of Albacete and VanLehn (2000) may be an effective means for learning.

The main focus of this study is on the knowledge transmission process and on the cognitive strategy used to shift teachers' instructional explanation of the concept of fraction multiplication and division from an instrumental to a relational understanding of mathematics. The development of the cognitive models for fraction multiplication and division was guided by the following conceptual process as shown below.

Pedagogical content knowledge (PCK) was characterized as the most regularly taught topics in one's subject area; the most useful forms of representation of those ideas; the most powerful analogies, illustrations, examples, explanations and demonstrations, including an understanding of what makes the learning of specific concepts easy or difficult and taking into consideration the schema that students of different ages and backgrounds bring with them to the learning process (Kinach, 2002; Reiman & Sprinthall, 1998; Kort & Reilly, 2002). According to Shulman (1987), PCK is a transformation process whereby prospective teachers' subject matter knowledge is converted into a form appropriate for teaching.

Mayer (1987) suggested that three major internal conditions must be met for instruction to foster meaningful learning. Instruction must help the learner to select relevant information, organize information and integrate information. Selection involves focusing attention on relevant pieces of the presented information and adding them to the short-term memory. Sternberg (1985) refers to this process as selective encoding and defines it as filtering of pertinent information. Organizing involves constructing internal connections among the incoming pieces of information into a coherent whole. Integrating involves constructing external connections between the newly organized knowledge to existing relevant knowledge to form an externally connected whole. Assessment must follow these three major internal conditions which involves seeking the consensus of experts

regarding the developed cognitive model (Wilson & Cole, 1996).

Cognitive Models Used

Fraction Multiplication Cognitive Model

An intensive review of the difficulties encountered by students in learning and the different approaches and models used to teach fraction multiplication, after which a multiplication cognitive model was restructured with the following main sub-goals: 1) identify the multiplicand and draw a pictorial representation of it using a rectangle with vertical divisions, 2) identify the multiplier and draw a rectangle representation of the same size with horizontal divisions, 3) superimpose the two representations and 4) represent the product using the double shaded regions as the numerator and the total number of regions made on the superimposed model as the denominator.

Fraction Division Cognitive Model

In the case of fraction division, students are traditionally

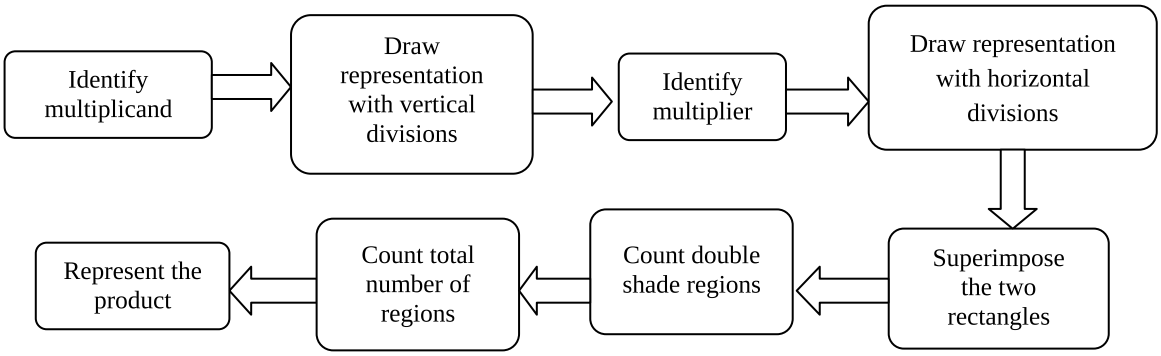
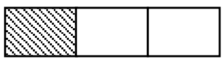
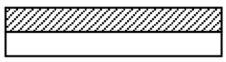
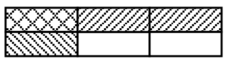
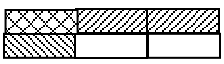


Figure 2. Fraction multiplication cognitive model

Table 2.
Fraction Multiplication Process using the Cognitive Model

Sub-goals	Prompter	Representation / Output
1. Identify the multiplicand	$\frac{1}{3} \times \frac{1}{2}$	$\frac{1}{3}$ is the multiplicand
2. Draw representation with vertical divisions	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify multiplier	$\frac{1}{3} \times \frac{1}{2}$	½ is the multiplier
4. Draw representation with horizontal divisions	Shade the portion representing ½ in a rectangular figure	
5. Superimpose the two rectangles		
6. Count double shaded regions (numerator)		There is only 1 double shaded region
7. Count total number of regions (denominator)		There is a total of 6 regions in the figure
8. Represent the product	1 as numerator and 6 as denominator	The product of $\frac{1}{3} \times \frac{1}{2}$ is $\frac{1}{6}$

taught the computational procedure of inverting the divisor and changing the operation to multiplication (invert and multiply strategy) (Meagher, 2002). Modifications on the picture division strategy (Jhonson & Koedinger, 2001; Tirosh, 2000; NCTM, 2000) were made to suite the kind of dividends and divisors, whether it is a whole number, a proper fraction or a mixed number. These confusions in regard to fraction representations made learning picture division quite difficult for students (Lubinski, C. & Fox, T., 1998).

The division cognitive model, adapting the ACT-R theory, will have its main sub-goals as follows: 1) identify the dividend and draw a pictorial representation of it using a number line or a rectangle, 2) identify the divisor and mark

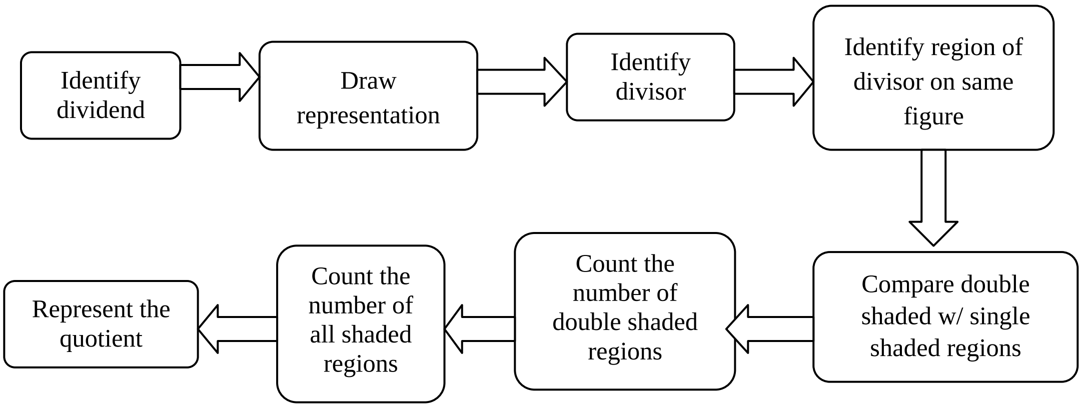
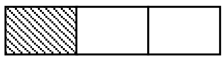
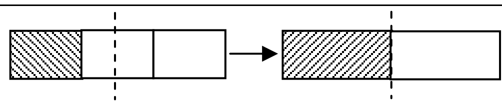
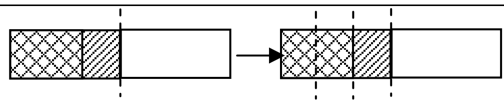
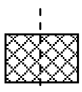
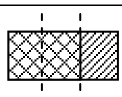


Figure 3. Fraction division cognitive model

Table 3.
Fraction Division Process using the Cognitive Model

Sub-goals	Prompter	Representation / Output
1. Identify the dividend	$\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{3}$ is the dividend
2. Draw representation	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify the divisor	$\frac{1}{3} \div \frac{1}{2}$	½ is the divisor
4. Identify the region of the divisor on the same figure	Shade the region representing ½ in the rectangular figure	
5. Superimpose and compare the double shaded with the single shaded regions	These regions must be of the same size	
6. Count the number of double shaded regions (numerator)		There are 2 double shaded regions in the figure
7. Count the number of all shaded regions (denominator)		There is a total of 3 shaded regions in the figure
8. Represent the quotient	2 as numerator and 3 as denominator	The quotient of $\frac{1}{3} \div \frac{1}{2}$ is $\frac{2}{3}$

the picture drawn according to the size of the divisor, 3) count the number of marked groups and consider this as whole number part of the quotient and 4) convert the remainder of the picture (if there is any) to a fraction and this will be the fractional part of the quotient.

Method

Subjects of the Study

Two sections, under the Bridge Program, in a public high school were purposively selected as control and experimental groups for the study. Students under the Bridge Program are those elementary graduates who did not pass the High School Readiness Test (HRST) and opted to be under the program. They are asked to take one more year of Math, Science and English to prepare them for secondary education.

Respondents had prior knowledge of the cancel-and-multiply and the invert-and-multiply algorithms as strategies for computing fraction multiplication and division in the lower grades and this knowledge was measured with the use of a pretest. The same lessons were repeated to them during the academic year the study was conducted as part of the math curriculum for the Bridge Program. Students in the control group underwent the traditional algorithmic way of teaching multiplication and division of fractions while the experimental group made use of the restructured cognitive models to understand the process of multiplying and dividing fractions and relate it to their schema of whole numbers. For the purpose of instruction, the teacher made use of the cut-out acetate form of fraction representations of the same rectangular size and shape in order to facilitate the superimposing of the needed fraction representations. After the lesson on fraction multiplication and division, another test *(posttest)* was given to them to find out how much they had learned.

Both pretest and posttest involved paper and pencil tests that consisted of 15 questions, of which seven of the questions were on multiplication of fractions, six were on division of fractions and two problem questions of each of these two fractional operations.. The posttest was parallel to the pre-test in terms of scope and level of difficulty. Students were asked to show their solutions in both tests.

Results

Pretest and posttest scores were analyzed in different ways.

Table 4 indicates that the mean pre-test score of the

Table 4

Comparison of Pre-test and Posttest Results

	Pretest		Posttest		*t*-value	*p*-value
	Mean	*SD*	Mean	*SD*		(2-tailed)
Control Group	1.76	1.27	5.60	3.39	5.80	*5.64E-06
Experimental Group	1.91	1.97	9.09	2.56	11.94	-8.04E-11
t-value	0.31		3.94			
p-value (2-talied)	0.76		*0.0003			

Note. *indicates significance difference at $\alpha \leq 0.05$

Table 5.

Comparison of Gain Scores from Pretest and Posttest of Control and Experimental Groups

	Mean Gain	*SD*	*t*-value	*p*-value(2-tailed)
Control Group	3.84	3.31	3.70	*0.0006
Experimental Group	7.18	2.82		

Note. *indicated significant difference at $\alpha \leq 0.01$

control group was 1.76 with a standard deviation of 1.27. On the other hand, mean pre-test score of the experimental group was 1.91 with a standard deviation of 1.97. No reliable difference was found between the two groups using t-test for independent samples assuming equal variances (t-value=0.31, p-value=0/76). This proves that the initial competencies of the two groups were equivalent.

To prove that both the algorithmic way of teaching fraction multiplication and division and the use of the cognitive models were both effective, their pretest and posttest mean results were compared with each other, respectively, using paired t-tests. The control group had a pre-test mean of 1.76 and a posttest mean of 5.60, showing a significant difference ($t = 5.80$, $p \leq 0.05$). The same is true with the experimental group. The pre-test mean of the experimental group was 1.91 and posttest mean was 9.09. There was a significant difference between the two tests ($t = 11.94$, $p \leq 0.01$). These results suggest that the two teaching methods under investigation were both effective in teaching fraction multiplication and division.

Their posttest mean scores were also compared. The control group had a posttest mean of 5.60 with a standard deviation of 3.39 and the experimental group had a posttest mean of 9.09 with a standard deviation of 2.56, indicating a significant difference ($t = 3.94$, $p \leq 0.01$). This indicates that students using the cognitive models gained higher scores on the posttest than those students relying solely on the algorithm.

Comparison of Gain Scores

To further prove the effectiveness of the cognitive models, the gain scores from pre-test to posttest were compared. The mean gain of the control group was 3.84 with a standard deviation of 3.31. On the other hand, the experimental group obtained a much higher mean gain of 7.18 with a standard deviation of 2.82. Using independent samples t-test, a reliable difference was found ($t = 3.70$, $p \leq 0.01$). This result suggests that the intervention of the cognitive models had a remarkable impact on the students' understanding of the concepts as well as on their ability to discard misconceptions.

Effect Size

Effect size is a standard way to compare the results of one pedagogical experiment to another. One way to calculate effect size, used in Albacete and VanLehn (2000), is to find the difference between the mean gain scores of the

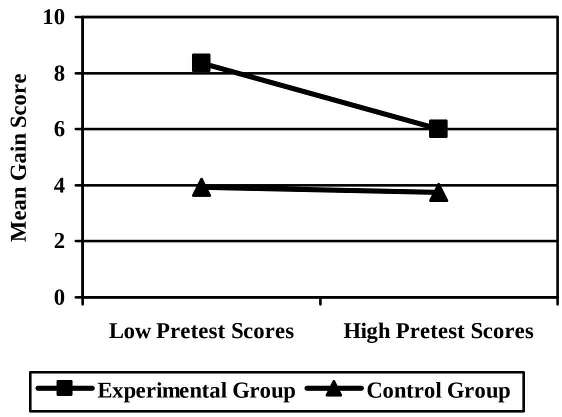


Figure 4. The Mean gain of the low and high pre-test score groups in the experimental and control groups

Table 6.

Comparison of Low and High Pre-test Score Group

		Control Group		Experimental Group	
		Mean	*SD*	Mean	*SD*
Low Pre-test Score Group	Pre-test	0.85	0.90	0.36	0.50
	Posttest	4.77	3.42	8.73	2.28
	Gain	3.92	3.62	8.36	2.46
High Pre-test Score Group	Pre-test	2.75	0.75	3.45	1.63
	Posttest	6.50	3.26	9.45	2.87
	Gain	3.75	3.11	6.00	2.76

experimental and control groups, and then divide it by the gain score standard deviation of the control group. The calculation yields (7.18 – 3.84)/3.31 = 1.01. This reflects a rather impressive effect on the learning of fraction multiplication and division when cognitive models are being utilized.

Innovative interventions such as the cognitive models, sometimes cause higher gains for students with higher pre-test scores. The intervention must be able to function for those students who need more help, as revealed by low pre-test scores. The experimental group was divided into 2 groups according to whether the student's pre-test score was above or below the median. The mean gain of the low pre-test score group was 8.36 with a standard deviation of 2.46 while the mean gain of the high pre-test score group was 6.00 with a standard deviation of 2.76. A statistically significant difference between the gain scores was found ($t = 2.12$, $p \leq 0.05$).

A similar analysis was done with the control group. The mean gain of the low pre-test score group was 3.92 with a standard deviation of 3.62, while the high pre-test score group' mean gain was 3.75, with a standard deviation of 3.11. No significant difference exist between the mean gain scores ($t = 0.12$, $p > 0.05$). Figure 4 illustrates these results:

It was interesting to note that in the experimental group the poorer students' knowledge gains (8.36) were significantly higher than those of the good students (6.00). In the experimental group, the lower gain score (6.00) in the high pre-test score group was not a consequence of a ceiling effect. With the highest possible score of 15 and a mean pre-test score of 3.45 for the high pre-test scoring subgroup of the experimental group, there is an opportunity for this group to have a gain score very close to that achieved by the group of poorer students. The highest posttest score obtained by the experimental group was 14 out of 15 items, while that of the control group as 13 out of 15 items. This indicated that there is one item in the posttest that contained knowledge which was difficult to grasp even with the use of the cognitive model.

Discussion

Implications of the Results of the Use of Cognitive Models

The empirical results revealed a remarkable advantage of using the cognitive models as an overture to teaching the algorithmic way of doing fraction multiplication and division. The use of cognitive models helped the students understand the algorithm better and relate it to their schema, thus achieving greater retention.

Students' errors in the experimental group, when using the cognitive models, indicated a need for more intensive ways of presenting division when the divisor is a mixed number. Otherwise, the models captured students' behaviors quite well.

Predictions from the Models

As in the study by Johnson and Koedinger (2001), the use of cognitive models and the use of algorithms in teaching fraction multiplication and division led to comparative predictions for the difficulty of learning each strategy, efficiency of using each strategy once learned, generality of each strategy on a range of fraction multiplication and division problems, retention of each strategy and transfer.

The ease of learning each strategy depends on students' prior knowledge. Learning difficulty can be predicted by two factors - how students represent the fractions and the fraction multiplication and division operations and how well they know symbol manipulation rules for working with fractions. On one hand, the algorithmic way of fraction multiplication and division became difficult to learn since the students represented fractions as visual arrangements of digits and the operation symbols without relation to their schema. On the other hand, learning through the cognitive models was relatively straightforward since a majority of the procedures were based on familiar and well-practiced knowledge.

The use of the cognitive models as a strategy to teach fraction multiplication and division proved to be more efficient once it is mastered than teaching only with algorithms. As proven by the gain scores of the respondents, the cognitive models were able to bridge the gap between students' schema of whole numbers and its operations and the cancellation and invert-and-multiply algorithm. The experimental group displayed more efficiency in their computations since they were able to infer from the models whenever they were not sure of their solution.

The ease of applying the two strategies to the full range

of fraction multiplication and division is not equivalent. Once the cancellation and invert-and-multiply algorithms are mastered, it can then be applied to any fraction multiplication and division problem. On the other hand, the use of cognitive models may become very cumbersome if the fractions are large, and this constraint becomes unmanageable.

The fourth prediction concerns retention of the concepts learned. Recall is based on spreading activation so knowledge that is connected to a richer network of knowledge chunks is easier to recall (Johnson & Koedinger, 2001). On one hand, the cognitive models utilize rich knowledge representations of quantities and operations, so this network of relations facilitate recall. On the other hand, the cancellation and invert-and-multiply strategies utilize sparse, visual based representations that are not connected to a rich knowledgebase so this strategy is harder to recall after some delay. Students had difficulty retrieving correctly all the relevant procedures for fraction multiplication and division algorithm. Thus, the use of cognitive models for these fraction operations will be more robust.

The strategies led to different transfer predictions. When knowledge chunks are activated, their memory trace of the individual is strengthened (Johnson & Koedinger, 2001). As students use these cognitive models, the quantity-based representation of fractions and a meaning-based representation of multiplication and division are strengthened and refined. Thus, it facilitated performance of tasks utilizing this representation. Representing fractions as part-whole quantities provided a basis for performing tasks such as partitioning, comparing magnitudes, estimating, adding and subtracting fractions. The cognitive models also provided meaning to the multiplication and division algorithm (the cancellation and invert-and-multiply strategies), giving reasons as to why the product and quotient may either be higher or lower than the factors, dividend or divisor. On the other hand, the cancellation and invert-and-multiply strategies facilitated computations on problems involving conversion of improper fractions to mixed numbers and vice versa, and the reduction of fractions and other algebraic expressions to their simplest form.

Using cognitive models as an aid to instruction led to more benefits than drawbacks. Its exploratory nature motivated the students to learn a tedious topic, such as fraction multiplication and division. Learning is achieved, not merely by telling the students the algorithm, but by the students themselves encountering the proofs of the algorithm as a result of their manipulation of the cognitive models. It proved to be easier to learn since the students have quantity-based representations of fractions, can be recalled after a delay, had been a more efficient strategy of multiplication and division once learned and was able to transfer to tasks comparing magnitudes, partitioning, estimating and doing operations such as addition and subtraction. It also provided greater meaning to fraction multiplication and division algorithms. On the other hand, the use of algorithms should be easy to learn if students are already clear in their conception of fractions and know how to manipulate them. It was efficient and broadly applicable once mastered and transferred the concept of fraction multiplication and division to algebraic computations. Parallel to the study made by Norris, Leighton, and Phillips (2004), cognitive models were found to theorize the content and capabilities of the students' minds, in terms of meta-cognition, reasoning strategies and principles of sound thinking.

Conclusion

Instruction should be able to bridge the gap from the more meaningful and grounded strategy of the use of cognitive models to the more abstract and efficient algorithms in order to maintain high retention. The cognitive models suggest a careful sequence of lessons for teaching the concept of fractions and the meaning of multiplication and division of fractions. Students should first learn to represent fractions as part-whole quantities. Next, they should be taught to use the cognitive models to find the product and quotient in fraction multiplication and division. Students must be able to see the meaning of multiplying and dividing with fractions so as to make sense of having a product or quotient which is higher or lower than any of the factors, dividend or divisor. They must be able to connect that the concept of fraction operations is not entirely different from that of doing the same operations with whole numbers.

The evaluation of the cognitive models suggests that the teaching strategy with the use of cognitive models for

developing target knowledge and handling misconceptions, is effective in accomplishing the task it was designed to perform. The experimental group surpassed the control group in every statistical test performed. The detailed examination of the effectiveness of each lesson with the use of the cognitive models showed a trend in favor of using these cognitive models.

Students pass through different levels of understanding in which mathematizing took place with the use of cognitive models, from devising informal context-connected solutions to reaching some level of schematization and finally to developing insights into the general principles behind a problem and being able to see the overall picture essential for learning. The cognitive model was able to assume its role of bridging the gap between the informal understanding connected to the real and imagined reality on the one hand and an understanding of a formal system of mathematical concepts on the other.

The teacher must be able to create cognitive models based on research-based models as well as their students' needs and characteristics that would assist them in making pedagogical decisions required of them on a day to day basis in the classroom. Teachers should be given opportunities to observe students engaged in mathematical activities, to interpret their responses and ways of thinking and experience the strengths and limitations of the various research-based models in their instruction.

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